Marginalia

What is random?

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I am convinced that the vast majority of my readers, and in fact the vast majority of scientists and even non-scientists, are convinced that they know what "random" is. A toss of a coin is random; so is a mutation, and so is the emission of an alpha particle. The motion of the planet Mercury (or any other planet) is not random, and neither is the propagation of an acoustical wave produced by a vibrating string. Simple, isn't it?

Well, not quite. If pressed for a definition of randomness, most people will fall back on such formulations as "lack of regularity" or "unpredictability." But they are then faced with the equally difficult task of defining "regularity" or "predictability," and soon find themselves immersed in metaphysics.

It would be impossible in such a short space to cover all the aspects of this subtle and fascinating subject, but I will take up some of the salient points, reserving others for a sequel to this discussion.

Let me begin with an example which will occupy most of our attention here. Suppose that a large number, m, of volunteers were asked to toss a coin 10,000 times each and to record the number of times, $N(H)$, the "heads" side appeared. We can then calculate the m numbers:

$$\frac{N(H) - 5{,}000}{50} \tag{1}$$

and construct a histogram based on them with interval size 0.5. In other words, for each interval $(0.5l, 0.5(l+1))$, $l = 0, \pm1, \pm2, \pm3, \ldots$, we calculate the percentage, p_l, of numbers (Eq. 1) that fall within this interval and then draw a horizontal segment over this interval at height p_l.

If the coins have not been tampered with in any way and if none of the tossers is a magician who can manipulate the outcome of the tosses, the resulting graph composed of adjacent intervals of heights p_l over the interval $(0.5l, 0.5(l+1))$ will be found to be well approximated by the so-called normal curve:

$$\frac{1}{\sqrt{2\pi}}\exp(-x^2/2)$$

Moreover, p_l will be well approximated by the area under the curve over the interval $(0.5l, 0.5(l+1))$, i.e.:

$$p_l \sim \frac{1}{\sqrt{2\pi}}\int_{0.5l}^{0.5(l+1)} \exp(-x^2/2)dx \tag{2}$$

This is an empirical fact, verified untold times, and the appearance of the normal curve has been and often still is taken as an indication, if not a proof, that the tosses were random.

Now if a coin is tossed 10,000 times, there are $2^{10{,}000}$ different outcomes, roughly 1 followed by a thousand zeros—a staggering number indeed! With each of these outcomes, one associates again the numbers (Eq. 1), of which there are only 10,000 different ones, and it is possible to compute in terms of binomial coefficients the percentage, q_l, of these numbers that fall within the interval $(0.5l, 0.5(l+1))$. One then verifies that Eq. 2 holds, with p_l replaced by q_l.

But now we are on the firm ground of mathematics, and in fact if 10,000 is replaced by n, 5,000 by $n/2$, and 50 by $\sqrt{n}/2$, then as n approaches infinity we have:

$$\lim_{n\to\infty} q_l(n) = \frac{1}{\sqrt{2\pi}}\int_{0.5l}^{0.5(l+1)} \exp(-x^2/2)dx \tag{3}$$

where $q_l(n)$ is the proportion (percentage) of numbers,

$$2\,\frac{N_n(H) - n/2}{\sqrt{n}}$$

that fall within the interval $(0.5l, 0.5(l+1))$. The statement embodied in Eq. 3 was proved by Abraham de Moivre in 1718 and is a very special case of what is now called the central limit theorem of the theory of probability.

The analogy between Eq. 3, which is a mathematical theorem, and Eq. 2, which is an empirical fact, is of course striking, and it led Henri Poincaré to the witty observation that all the world believes in the normal curve "car les experimenteurs s'imaginent que c'est un théorème de mathématiques et les mathématiciens que c'est un fait experimental" (*1*).

The ubiquity of the normal curve in the world of phenomena associated with randomness and chance has made it a veritable symbol of this world. Whenever and wherever the normal curve was encountered, a suspicion at once formed that chance—the elusive, the capricious, the unpredictable—was at play. But what about Eq. 3, which is as much a "théorème de mathématiques" as is the Pythagorean theorem? The answer runs something like this: Eq. 3 is a statement about the *totality* of possible outcomes, while Eq. 2 is about a representative sample of this totality; to choose a truly representative sample, one needs a random mechanism, which in our case is provided by m people tossing m "honest" coins.

Everybody knows, of course, that choosing representative (or random, as they are called) samples is big business, and that sampling is not carried out by any "random mechanism." It is usually accomplished by combining some kind of stratification with computer programs that generate what we now call "pseudo-random numbers." The very term "pseudo-random" implies a belief that there are also "truly random" numbers, presumably produced by "truly random" mechanisms (e.g., people tossing coins). If all this sounds vague and "soft," it is because it *is* vague and "soft." The only sensible question that can be raised is whether it is possible to distinguish results obtained by what are considered to be truly random mechanisms from those produced by appropriately contrived, purely deter-

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ministic ones such as computer programs.

To be specific, let us go back to coin tossing. First consider a number x chosen from the interval (0,1) and the numbers $2x$, 2^2x, 2^3x, ... Consider further the fractional parts $f_1, f_2, f_3, \ldots$, of these numbers; whenever f is less than $1/2$ write H, and whenever it is greater than (or equal to) $1/2$ write T (2). In this way each x generates an infinite sequence of H's and T's. Suppose now that someone tosses a coin, recording each time whether it lands showing "heads" (H) or "tails" (T). He too will obtain a sequence of H's and T's, and the question becomes: If one is given a sequence of H's and T's, can one tell whether it was produced by tossing or by the arithmetical process based on numbers $2x$, 2^2x, 2^3x, . . . , for an appropriately chosen x? Well, one cannot. No matter what tests of randomness are applied, the arithmetical sequence will pass them, unless one has been singularly unlucky in the choice of x.

For example, if a coin is tossed $m \times 10{,}000$ times, one can subdivide the sequence of results into m subsequences of consecutive tosses, each of length 10,000, and construct a histogram, as explained earlier. This histogram is likely to be well approximated by the normal curve, which will be taken as evidence that we are dealing with a random process. But if we do the same thing to the data obtained by the arithmetical process, then the set of x's that produce histograms not well approximated by the normal curve will turn out to be small (in a technical sense that will not be explained here) and hence it is not likely that an x will be picked from such a set.

I am being vague in using the term "well approximated," but I do this only to avoid technicalities which might obscure the exposition. The reader will have to take my word for the fact that the term can be made precise. A more serious objection can be raised to my use of the word "unlucky," and to the implication that since exceptional x's belong to a "small" set they are unlikely to be chosen. This smacks of trying to sneak elements of chance and randomness into a discussion of a deterministic process, and indeed the objection is well taken.

Rational x's—x's that are common fractions—have to be excluded, because they lead to sequences with obvious periodicities and would be rejected at once as being nonrandom. Even irrational x's can produce regularities or yield histograms that do not approximate the normal curve well. But neither is there a guarantee that real tosses will not exhibit regularities or that they will always present us with acceptable histograms. Regularities are in the eye of the beholder, and the annals of science are filled with spurious discoveries based on spotting something striking in the data.

Let me then state categorically that there is no way to tell with any degree of confidence whether a sequence of H's and T's has been generated by tosses of a coin or by an arithmetical procedure based on an appropriately chosen x. Even if a sequence consists of all H's (or all T's), it could have been generated by real tosses of a normal coin. It could be even more simply generated by the arithmetical process. Of course, if time after time in tossing a coin only H's (or only T's) appear, it would be advisable to take a careful look at the coin and to find out something about the fellow who is doing the tossing. You may find out that you are being taken for a ride. From the purely *operational* point of view, however, the concept of randomness is so elusive as to cease to be viable.

Distressing as this conclusion may be to some, I have one more thing in store for them. In his book *Chance and Necessity* (3), the late Jacques Monod suggests that a distinction should be made between "disciplined" chance, as used, for example, in physics, and "blind" chance, exemplified by the death of a doctor who was killed on his way to see a patient by a brick that fell from the roof of a building. There are no statistics, as far as I know, on doctors being killed by bricks falling from the roofs of buildings, but there are extensive statistics on Prussian soldiers being kicked to death by horses (4). These unfortunate events clearly fall within the category of blind chance, and yet the data conform to the same Poisson law as do the data on the emission of alpha particles.

To be more precise, the proportion of consecutive time intervals of duration τ (e.g., a week) during which k soldiers are killed by horse kicks is approximately $\exp(-a\tau)(a\tau)^k/k!$. Similarly, the proportion of consecutive time intervals of duration t (e.g., one second) during which k particles are emitted is approximately $\exp(-\alpha t)(\alpha t)^k/k!$. Here a and α are empirically determined parameters, although in principle α may also be calculable theoretically. By a proper adjustment of units (amounting to setting $a\tau = \alpha t$) the two sets of data (one on soldiers killed, the other on alpha particles) will be difficult to distinguish. They are certainly indistinguishable on the basis of distribution (they both follow identical Poisson laws), which is the first test one would use. I do not know what other tests could be applied, but I am convinced that whatever they are, the results would be at best inconclusive.

Let me end on a reassuring note. The discussion of randomness belongs to the foundations of statistical methodology and its applicability to empirical sciences. Fortunately, the upper reaches of science are as insensitive to such basic questions as they are to all sorts of other philosophical concerns. Therefore, whatever your views and beliefs on randomness—and they are more likely than not untenable—no great harm will come to you. If the discipline you practice is sufficiently robust, it contains enough checks and balances to keep you from committing errors that might come from the mistaken belief that you *really* know what "random" is.

References

1. H. Poincaré. 1912. *Calcul des Probabilités*, 2nd ed. Paris: Gauthier-Villars. ("All the world believes in the normal curve, because the experimental scientists imagine that it is a mathematical theorem and the mathematicians imagine that it is something experimental.")
2. The study of fractional parts of $2^n x$, $n = 1, 2, 3, \ldots$, and their connection with the mathematical theory of coin tossing has a long history. The latest discussion is found in an interesting article by J. Ford, 1983, How random is a coin toss?, *Physics Today*, April, pp. 40–47. For further details see my 1959 monograph, *Statistical Independence in Probability Analysis and Number Theory*, Wiley.
3. J. Monod. 1971. *Chance and Necessity*. (Trans. from the French by A. Wainhouse.) Knopf.
4. These data were first published and discussed by L. von Bortkiewicz. See F. A. Haight, 1967, *Handbook of the Poisson Distributions*, Wiley.