Mathematical

Games

Patterns in primes are a clue to the strong law of small numbers

by Martin Gardner

Let us now praise prime numbers
With our fathers that begat us:
The power, the peculiar glory
of prime numbers
Is that nothing begat them,
No ancestors, no factors,
Adams among the multiplied generations.

—HELEN SPALDING

The Strong Law of Small Numbers is the provocative title of an unpublished paper by Richard Kenneth Guy, a mathematician at the University of Calgary. For many years Guy has edited the "Research Problems" department of The American Mathematical Monthly. He is the author of numerous technical papers and is coauthor with John Horton Conway and Elwyn R. Berlekamp of Winning Ways, a long-awaited mammoth book of new mathematical recreations that will be published by Academic Press this coming year. The material that follows is taken almost entirely from Guy's paper.

"We think of mathematics as an exact science," Guy begins, "but in the field of discovery this is not at all the right picture. Two of the most important elements in mathematical research are asking the right questions and recognizing patterns."

Unfortunately there is no procedure for generating good questions and no way of knowing whether an observed pattern will lead to a significant new theorem or whether the pattern is just a lucky coincidence. In these respects the research mathematician is in a position strangely like that of the scientist. Both ask questions, do experiments and observe patterns. Will an observed pattern be repeated when new observations are made, with new parameters, leading to the discovery of a general law, or will counterexamples turn up that contradict a hypothesis? Is it true that mathematicians can do something scientists cannot: they can prove theorems within a formal system. Until a proof is found, however, a mathematician relies on fallible empirical induction in much the same way a scientist does. This is particularly true in combinatorial problems that involve infinite sequences of numbers.

In examining cases involving small numbers a striking pattern may be encountered that strongly implies a general theorem. It is this implication Guy calls the strong law of small numbers. Sometimes the law works, sometimes it does not. If the pattern is no more than a set of coincidences, as it often is, a mathematician can waste an enormous amount of time trying to prove a false theorem. The law can also mislead in an opposite way. A few counterexamples may cause the mathematician to prematurely abandon a search for a theorem that is actually true but slightly more complicated than expected.

Today's computers are a big help because they often can quickly explore cases of higher numbers that will either explode a hypothesis or greatly increase the probability of its being true. In many combinatorial problems, however, the numbers grow at such a fantastic rate that the computer can examine only a few more cases than can be examined by hand, and the mathematician may be left with an extremely intractable problem.

One could fill many books with examples that follow we shall limit our efforts to prove them. In the ragbag of results that may be there but resist all efforts to prove them. In the ragbag of phenomena there will be found many more cases than can be examined by hand, and the mathematician may be left with an extremely intractable problem.

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In the beginning where chaos Ends and zero resolves,
They crowd the foreground prodigal
As forest,
But middle distance thins them,
Far distance to infinity
Yields them rarely as unreturning comets.

Primes offer rich examples of remarkable patterns that are entirely accidental and lead nowhere. Consider the following sequence of primes: 7, 37, 337, 3337, 33337, 333337,... If the sequence is self-elected, the pattern will continue, but it fails in the next case: 3333337 is prime.

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Several years ago, Ru F. Fortune, an anthropologist at the University of Cambridge (who was once married to the late Margaret Mead), noted a curious pattern involving small primes. Starting with 3, take the product of a set of consecutive primes. Add 1. Find the
Gilbreath’s conjecture concerning successive absolute differences in the sequence of primes

next-largest prime and from it subtract the product of the consecutive primes. Is the result always a prime? The chart on page 18 shows the procedure applied to the first eight cases and gives the eight “fortunate primes” that are generated.

Fortune conjectures that the result is always a prime. Most number theorists believe this is true, but no proof has been found and there seems to be little hope, Guy says, of finding one in the foreseeable future. Perhaps some reader of this column can “cook” (falsify) the conjecture by finding what one might call a “fortune cookie.” Note that in the chart the first five numbers at the right side of the equation on the left are primes. Is this always the case? No, it fails for the next three numbers. Mark Templer in his article titled “Primality of \( k! \)” has shown that one more than a prime conjectures is that there are an infinite number of twin primes.

The twin-prime conjecture generalizes to prime pairs that differ by any “reasonable” size. For example, the following triplets of primes all fit the pattern \( 2 \times 3, 7, 11 \) and \( 13, 17, 19 \) and \( 29, 31 \). The law fails for \( 3, 5, 7 \) and \( 11, 13 \). No one knows if the number of Mersenne primes is infinite, or even if there is a 28th one. Mersenne number be prime? The strong law of small numbers suggests it will, because it is true when \( n = 2 \) and 7. The law fails for \( n = 11 \), however, because \( 2^{11} - 1 \) equals 2047, which equals \( 23 \times 89 \). It holds for \( n = 13, n = 17 \) and \( n = 19 \), then fails again for \( n = 23 \). From here on successively larger entries become very rare. At the moment only 27 Mersenne primes (hence only 27 perfect numbers) are known. The 27th Mersenne prime, \( 2^{242497} - 1 \), was discovered in 1979 by a computer program written by David Slowinski with the assistance of Harry L. Nelson at the Lawrence Livermore Laboratory of the University of California. The number starts 854... ends 671 and has 13,395 digits.

No one knows if the number of Mersenne primes is infinite, or even if there is a 28th one.

Fermat numbers have the form \( 2^{2^n} + 1 \). For \( n = 0, n = 1, n = 2, n = 3 \) and \( n = 4 \) the number is prime (3, 5, 17, 257 and 65537). Pierre de Fermat thought all numbers of this form are prime, but he overlooked the fact that \( n = 5 \) yields 4294967297, which has the prime factors 641 x 6700417. No Fermat primes other than the five known to Fermat have been found, and no one knows whether or not others exist.

Here is a curious pattern involving factorial primes. Factorial \( n! \), written \( n \times (n-1)! \) or \( n! \), note
How long does this continue to give primes?

<table>
<thead>
<tr>
<th>n</th>
<th>SEQUENCE</th>
<th>k</th>
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</thead>
<tbody>
<tr>
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<td>1</td>
<td>2</td>
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<tr>
<td>2</td>
<td>1, 2, 1</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>1, 3, 2, 3, 1</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>1, 4, 3, 2, 3, 4, 1</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>1, 5, 4, 3, 2, 5, 3, 4, 5, 1</td>
<td>11</td>
</tr>
<tr>
<td>6</td>
<td>1, 6, 5, 4, 3, 2, 5, 3, 4, 5, 6, 1</td>
<td>13</td>
</tr>
<tr>
<td>7</td>
<td>1, 7, 6, 5, 4, 3, 7, 5, 2, 7, 5, 3, 7, 4, 5, 6, 7, 1</td>
<td>19</td>
</tr>
</tbody>
</table>

Leo Moser’s prime triangle

In each case the number on the right is prime. Alas, the strong law of small numbers fails on the next step. It yields

326981, the product of primes 79 and 83.

4139. The next primes result when we continue this manner, bringing each prime down as the first number of the next row, and adding numbers from the sequence 2, 4, 6, 8, ... In every case on the chart the result is a prime. Does this success continue forever or does it fail at some point? I shall give the answer in my next column.

The Canadian mathematician Leo Moser constructed the curiosity displayed in the illustration above. A study of the pattern shows that each sequence is formed from the one above it by inverting n, the row number, between all pairs of numbers that add to n. On the right k stands for the number of pairs in each sequence. Note that the first six k numbers are the first six primes. The next k number skips 17, but 19 is a prime. Are all k numbers prime? What is the formula for finding the n-th k number? I shall give the answers to both questions in my next column.

Except for 2, all primes have the form 4k + 1, which means that every prime except 2 is one more or one less than a multiple of 4. (This follows trivially from the fact that every odd number is one more or one less than a multiple of 4.) Write the odd primes in consecutive order, putting the 4k + 1 primes in the top row and the 4k + 1 primes under them:

3 7 11 19 23 31 43 47 53 61 73

At this point the top row is “winning the race.” If we continue the two sequences, will the top row always be ahead? You should not waste time trying to settle this empirically, Guy advises, because you have to go a long way before the second row gets ahead, and even then you will not have proved anything. The eminent Cambridge mathematician John E. Littlewood showed that the rows alternately lead infinitely often.

Above 5 all primes have the form 6k ± 1. If we race these two “horses,” they too change lead infinitely often. Other prime-number races have been investigated, such as the four horses in the 8k ± 1, 8k ± 3 race. Although it is far from established, most number theorists believe that in all such races, regardless of the number of horses, every horse is ahead infinitely often in the long run.

Primes of the form 4k + 1 (the bottom row of the 4k ± 1 race) can always be expressed as the sum of a unique pair of distinct square numbers. Hence 5 equals 4 + 1, 13 equals 4 + 9, and so on. This was proved by Fermat and is known as Fermat’s two-square theorem. It is an excellent example of a pattern for which the strong law of small numbers is not deceptive but leads to a genuine theorem. Many ways to prove the theorem have long been known, but in 1977 Loren C. Larson of St. Olaf Col-
Casio's Mariner

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College in Minnesota published a delightful new proof based on the familiar problem of placing \( n \) queens on an \( n \times n \) chessboard so that no queen attacks another.

The figure at the top of the illustration below shows the smallest solution for the queens problem that displays the following properties: (1) there is a queen on the center square, (2) all other queens are reached from the center by a generalized knight move of \( m \) cells in one direction followed by \( n \) cells in a direction at right angles to the first (where \( m \) and \( n \) are distinct integers) and (3) the final pattern has fourfold rotational symmetry (is unchanged by 90-degree rotations). The next-largest solution with all these features is shown at the bottom of the illustration: 13 queens on a 13-by-13 board.

Apart from the center queen, for all such solutions each quadrant of the board obviously must hold the same number of queens. The number therefore will have the form \( 4k + 1 \). Larson shows that solutions of this type can be constructed if and only if the number of queens is a prime of this form.

In all such solutions the board can be divided into identical smaller squares in the manner shown by the slanting lines in the two examples below. If we imagine the board formed into a torus by joining the top and bottom edges and the left and right edges, we see that each

Chessboard proof that \( 4k + 1 \) primes are the sum of two distinct squares.
board of side \( p \) is made up of \( p \) tilted squares. Since the board has an area of \( p^2 \), the area of each small square is \( \sqrt{p} \). Since \( p \) is the hypotenuse of a right triangle with sides equal to \( m \) and \( n \) (the two components of the generalized knight move), it follows from the Pythagorean theorem that \( p \) (the area of the square on the hypotenuse) must equal the sum of the squares of \( m \) and \( n \). And since \( p \) is any prime of the form \( 4k + 1 \), it follows that every such prime is the sum of two distinct squares. I have given Larson’s proof, based on earlier work by George Pólya, in highly abbreviated form. For more details see his article “A Theorem about Primes Proved on a Chessboard” (Mathematics Magazine, Vol. 50, No. 2, pages 69-74; March, 1977).

The fourth and last stanza of Spalding’s poem gives a fitting conclusion:

O prime improbable numbers,
Long may formula-hunters
Steam in abstraction, waste
to skeleton patience:
Stay non-conformist, nuisance,
Phenomena irreducible
To system, sequence, pattern
or explanation.

My description in September of my visit with Dr. Matrix contained a number of errors. I misspelled “Istiklal,” a street in Istanbul, and when I referred to the “Asian sector” of the city, I really meant the Old City. Dr. Matrix’ parting words to me were simply “Gule gule,” literally “Go with laughter.” I am indebted to Robert F. Scott, Boris Gilode and George Gibson, who were the first to write me in this connection.

There were also two mathematical errors. I reported Dr. Matrix as saying that the set of no-rep emirps (primes without duplicate digits that are different primes when they are written backward) had not been proved finite. Consulting my notes, I see that Dr. Matrix spoke only of the set of emirps. It was foolish of me not to realize that any emirp of more than 10 digits would have duplicate digits. Moreover, as Harvey P. Dale and many others noted, no number of even 10 digits can be a no-rep emirp because the 10 digits will have a sum of 45, and so any permutation of them will be a multiple of 9. According to Dale, the highest no-rep emirp is 987653201. I should not have listed 11939 as a cyclic emirp because the permutation 19391 is a palindrome. It appears as if the only cyclic emirp is the six-digit one Dr. Matrix gave, although it may never be possible to prove it. As John Baum made clear, my memory of Dr. Matrix’ remark about a recent paper by Paul Erdős was faulty. Erdős’ finding concerns the number 70, but the number whose property I described is 60. What Erdős actually proved was that if you start with \( n \) and write a sequence of integers greater than \( n \) such that each is relative-prime to all the preceding integers in the sequence, then 70 is the largest number for which such a sequence contains only primes and squares of primes.

In October, when I gave the answer to Dr. Matrix’ question about cutting a cube into three congruent parts, I said I knew of no other way to accomplish the trisection. As many readers were quick to point out, I could not have been wronger. John E. Morse sent the most general solution. If you hold a cube so that its corner points toward you and the outline appears to be a regular hexagon, you will see the cube’s 3-symmetry. This symmetry is too complicated to explain in detail, but it makes it possible to slice the cube into three congruent parts in an infinite number of ways. The surfaces of the parts may be flat or curved in any manner, and it is easy to design weird trisections for which the parts are so intricately interlocked that they cannot be separated.

Michael O’Donnell told me that the ballad about Abdul Abulbul Amir had been written in 1877 by Percy French, an Irish music-hall entertainer. A London publisher printed the song without acknowledgment, and to this day its author is listed in anthologies as Anonymous. For the best version I know see Sigmund Spaeth’s 1926 anthology Read ‘em and Weep—the Songs You Forgot to Remember.

Bennett Battaile wrote to suggest an alternative to the problem of labeling the corners of a cube with the integers 0 through 7 so that the sum of every pair of numbers sharing an edge is a prime. Try to make each sum a composite number. This problem too turns out to have a unique solution, apart from rotations and reflections.

The solution to last month’s problem of constructing a taxicab parabola with a given focus \( A \) and a given directrix is shown in the illustration above.