GEOMETRY

7

CURVES OF CONSTANT WIDTH

This section is concerned with an interesting class of convex curves, the so-called curves of *constant width*. These curves are also considered in Chapter 25 of an excellent elementary book by H. Rademacher and O. Toeplitz,† which can be used to supplement substantially the present section.

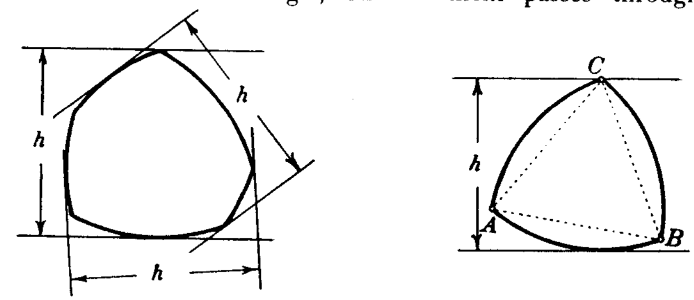
The exercises of the first half of this section (up to and including Exercise 7-14) are not related to the material of the preceding sections (except Section 1); a previous result is essential only for the solution of Exercise 7-12, namely Exercise 3-4 a. If Section 7 is read before Section 3, then in particular one needs only to acquaint himself with Exercise 3-4 a (by simply reading through the solution). Exercises 7-15 through 7-18 assume knowledge of the material explained in Exercises 4-1 through 4-11. Exercises 7-17 and 7-18 depend in an essential way upon Exercise 6-9. If Section 7 is read before Section 6, then these exercises must be omitted. Finally, Section 4 must be understood in order to follow the concluding part (in fine print) of this section.

The *width of a convex curve in a given direction* is the distance between a pair of supporting lines of the curve perpendicular to this direction (see Diagram 68, page 46). If the width of a curve is the same in all directions, then it is called a curve of *constant width* (Diagram 91 a). For such a curve we simply speak of its *width* instead of its width in a given direction.

The simplest example of a curve of constant width is a circle. Its width equals its diameter. However there are, besides the circle, infinitely

† *Von Zahlen und Figuren*, tr. by H. S. Zuckerman under the title *The Enjoyment of Mathematics*, Princeton University Press, 1957.

many other curves of constant width. For example, consider an equilateral triangle ABC of side h, and let each two of its vertices be joined by a circular arc of radius h, whose center is at the third vertex (Diagram 91 b). This curve is called a *Reuleaux triangle.* Given any two parallel supporting lines of a Reuleaux triangle, one of them passes through some



a Diagram 91 b

vertex of the triangle ABC (these are also corner points of the curve), while the other is tangent to the opposite circular arc. Hence the distance between two parallel supporting lines of a Reuleaux triangle is h.

Beginning with Reuleaux triangles, it is easy to find other examples of curves of constant width. Consider again an equilateral triangle with side of length h. About each vertex of the triangle let us draw inside the corresponding angle an arc of radius $\rho > h$; join the end points of the resulting three arcs by smaller arcs of radius $\rho' = \rho - h$ about the vertices of the triangle (Diagram 92), and let $\rho + \rho' = H$. Given any two parallel supporting lines of the resulting curve, one is tangent to an arc of the larger circle and the other to an arc of the smaller circle, and both arcs have the same center. Thus it is evident that this curve has constant width H.

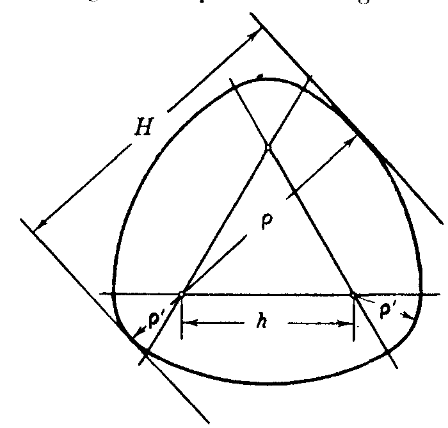
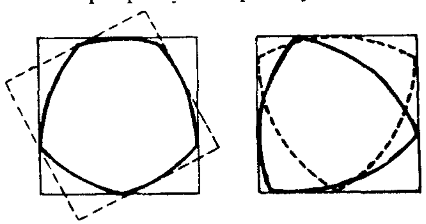


Diagram 92

If we draw two pairs of parallel supporting lines to a curve of constant width h so that the lines of the one pair are perpendicular to the lines of the other, then we obtain a square of side h. In this way a square can be circumscribed about a curve of constant width so that a side of the circumscribed square has any desired direction

(Diagram 93 b). In other words, the square can be rotated as much as desired and remains always circumscribed about the given curve of constant width. Or, a curve of constant width can be freely rotated within square so that it always maintains contact with the sides of the square the (remains inscribed in the square; see Diagram 93 b). It is clear that this property completely characterizes curves of constant width.



a b
Diagram 93

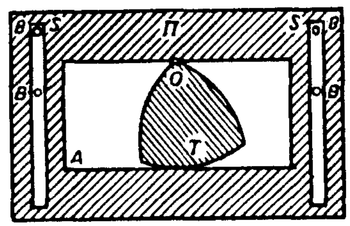
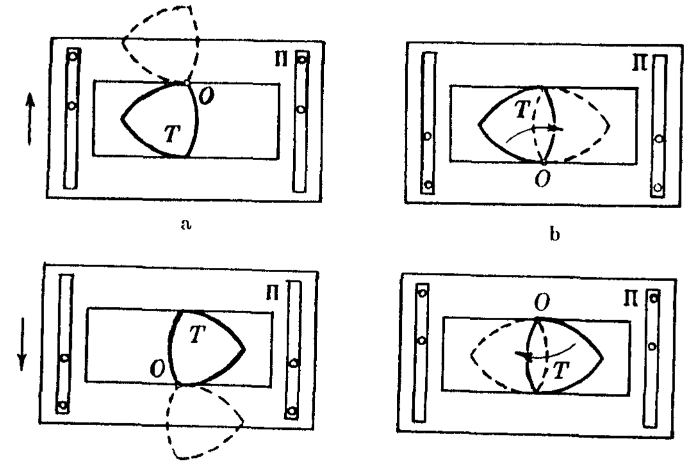


Diagram 94

Curves of constant width, especially the Reuleaux triangle, are the simplest curves of this type except for the circle, and are used in practice in various mechanisms. For example, the fact that a Reuleaux triangle can be freely turned within a square while it touches all the sides is used in boring square holes, and borers in the shape of a Reuleaux triangle are used for this purpose.

Consider still another example. Let Π be a plane plate with two vertical slits S and a rectangular opening A (Diagram 94). In each of the slits S there are two bolts B, which are fastened to an immovable base so that the plate Π can move freely up and down but not sideways. In the opening A there is a disc T in the form of a Reuleaux triangle which is fastened to an axis O perpendicular to the plane of the sketch. We cause the disc T to move about the axis O and observe the motion of the plate Π. It is easy to see that when T moves through 120° the plate Π is raised (Diagram 95 a); movement of the disc through 60° more causes no movement of the plate (Diagram 95 b); on moving the disc through the next 120°, the plate drops to its initial position (Diagram 95 c); movement through the last 60°, which constitutes the conclusion of a rotation, leaves the plate again motionless (Diagram 95 c). In this fashion the rotation of the disc T is changed into a linear motion of the plate Π. The fact that each interval of movement of the plate is followed by an equal interval of rest permits the application of this mechanism as a gripper for moving the film in a movie projector.†

† To avoid blurring of the image, the film in a movie projector must move intermittently; that is, the movement of the film (objective closed) must alternate with temporary motionlessness of the film (objective open). The mechanism which makes this movement of the film possible is called a *gripper*.



a b c d
Diagram 95

7-1. Calculate the perimeter of a Reuleaux triangle and its area. Which is greater, the area of a circle or the area of a Reuleaux triangle of equal width? Also determine the size of the interior angles at the corner points of a Reuleaux triangle.

7-2. Draw a curve of constant width h which consists of five, seven, or in general, of any arbitrary odd number of circular arcs of radius h. What is the length of each of these curves?

Curves of constant width h which consist of circular arcs of diameter h play an important role in the succeeding developments (*cf.* the text in small type, pages 76-80).

7-3. Prove that the distance between two points of a curve of constant width h cannot exceed h.

7-4. Prove that each supporting line has only one point in common with a curve of constant width h. In a curve of constant width, each chord joining the contact points of two parallel supporting lines is perpendicular to those lines and hence has length h.

Plane curves with the property that the distance between any two points of the curve does not exceed a certain quantity h, and such that for each point of the curve there exists another point of the curve at a distance h from it, are occasionally called *curves of constant diameter*. From Exercise 7-4 it follows that *every curve of constant width is also a curve of constant diameter*. It is easy to prove that conversely *each curve of constant diameter is also a curve of constant width*. (Prove this!) Hence we need not study curves of constant diameter any further; they are identical with curves of constant width.

We shall call the chords described in Exercise 7-4 the *diameters* of a curve of constant width.

7-5. Prove that each chord of a curve of constant width whose length equals the width of the curve must be a diameter.

7-6. Prove that any two diameter of a curve of constant width must intersect in the interior or on the curve. If they intersect on the curve, then their point of intersection A is a corner point of the curve, and the exterior angle of the curve (see Section 1, page 12) at A is not less than the angle between the two diameters.

7-7. Prove that the circle is the only curve of constant width with a center of symmetry.

7-8. Prove that if a curve of constant width h has a corner point, then one of the arcs of the curve is a circular arc of radius h.

Conversely, if some arc of a curve of constant width h is an arc of a circle of radius h, then the curve has a corner point.

7-9. Prove that the interior angle at a corner point A of a curve of constant width cannot be less than 120°. The only curve of constant width in which a corner has an interior angle of 120° is a Reuleaux triangle.

7-10. (Lemma) Let $ABCD$ be a rhombus, and let MN and PQ be two line segments which are perpendicular to the diagonal BD and whose distance apart is h (Diagram 96).

(a) Prove that the perimeter of the hexagon $AMNCQP$ does not depend on the position of MN and PQ.

(b) Prove that the area of the hexagon $AMNCQP$ assumes its maximum value when MN and PQ are at an equal distance $h/2$ from the diagonal AC of the rhombus; the area is a minimum when MN passes through the vertex B (or PQ passes through the vertex D) of the rhombus.

7-11. By examining equiangular polygons of 2^n sides which are circumscribed about an arbitrary curve K of constant width h and also about a circle of diameter h, derive Barbier's Theorem: *All curves of constant width h have length πh.*

7-12.* Examine equiangular polygons with $3 \cdot 2^n$ sides which are circumscribed about an arbitrary curve K of constant width h, about a circle of diameter h, and about a Reuleaux triangle T of width h. Prove that the circle has the greatest area, and the Reuleaux triangle the least, of all curves of constant width.

From Barbier's Theorem (Exercise 7-11), it follows that the theorem in the first part of Exercise 7-12 is also a consequence of the isoperimetric problem (see Exercise 5-8).

7-13. Given any convex curve of constant width, prove that the circle inscribed in K and that circumscribed about K must be concentric, and that the sum of their radii equals the width of the curve (Diagram 97).

7-14.* Prove that the Reuleaux triangle is the curve of constant width h with the greatest circumradius (and hence with the least in-radius; see Exercise 7-13). The circle, on the other hand, has the smallest circumradius (and the greatest in-radius).

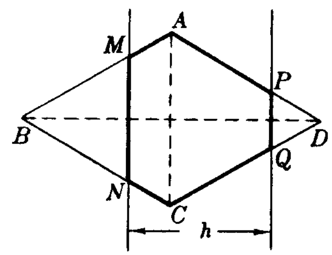


Diagram 96

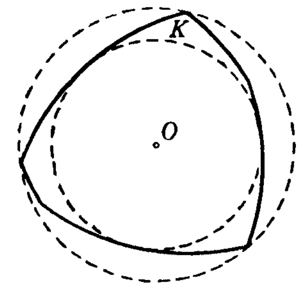


Diagram 97

In problems about curves of constant width the notion of the addition of convex curves (see Section 4) turns out to be quite useful. It follows at once from Exercise 4-11 that the sum of two curves of constant width is also a curve of constant width; this theorem allows us to construct from any curves of constant width new examples of such curves. We have used this fact already: as an example of a curve of constant width, we considered the sum of a Reuleaux triangle and a circle (Diagram 92).

7-15. Prove that the sum of an arbitrary curve of constant width h with the same curve turned through 180° is a circle of radius h. Obtain from this theorem a new proof of Barbier's Theorem (see Exercise 7-11).

7-16. (Converse of Exercise 7-15.) If the sum of a curve K with the curve K' obtained by rotating K through 180° is a circle, then K is a curve of constant width.

From Exercises 7-15 and 7-16 it follows that we can also define a curve of constant width as *a curve which yields a circle when it is added to the curve obtained by rotating it through 180°.* All the properties of curves of constant width are easily derived from this definition.

7-17. (a) Prove that of all convex curves of diameter 1 (see page 9), curves of constant width 1 have the greatest length.

(b) In a curve K of constant width D, let AB and PQ be two parallel chords such that the diagonals AQ and BP of the trapezoid $ABQP$ are diameters of the curve (Diagram 98). We denote the distance between the lines AB and PQ by Δ. Prove that the curve K, formed from the arcs AP and BQ of K and the chords AB and PQ has the greatest length of all convex curves of diameter D and width Δ (see pages 9 and 18).

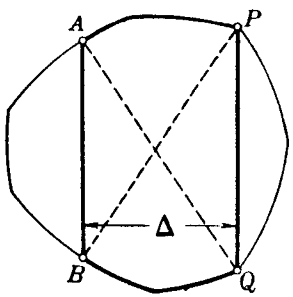


Diagram 98

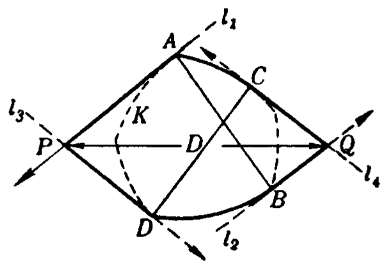


Diagram 99

7-18. (a) Prove that curves of constant width 1 have the least length of all convex curves of width 1.

(b) Let AB and CD be two diameters of a curve K of constant width Δ; let l_1 and l_2 be two supporting lines of K perpendicular to AB, and let l_3 and l_4 be supporting lines perpendicular to CD. Denote by P the intersection of l_1 and l_3, and by Q the intersection of l_2 and l_4 (Diagram 99). Let D be the distance between the points P and Q. Prove that the curve K shown in Diagram 99, formed from the segments AP, PD, BQ, QC and the arcs AC and BD of K has the smallest length of all curves of width Δ and diameter D.

From Exercises 7-17 and 7-18 we obtain the following estimates for the length L of a convex curve K of diameter D and width Δ:

$$\pi D \geq L, L \geq \pi\Delta; \qquad 2[D(\pi/2 - \text{arc cos } \Delta/D) + \sqrt{H^2 - h^2}]$$
$$\geq L \geq 2[\Delta(\pi/2 - \text{arc cos } \Delta/D) + \sqrt{D^2 - \Delta^2}].$$

(Here H and h are respectively the maximum and minimum altitudes among all the altitudes of equilateral triangles circumscribed about K.)

We now prove: *If K is any curve of constant width h, a curve of constant width h can be constructed which consists of circular arcs of radius h and which is arbitrarily close to K* (in the sense of the distance between curves defined in Section 4).

To prove this, consider a polygon $A_1A_2 \dots A_{2n}$ circumscribed about K, having an even number of sides, whose opposite sides A_kA_{k+1} and $A_{n+k}A_{n+k+1}$ ($k = 1, 2, \dots, n-1$), A_nA_{n+1} and $A_{2n}A_1$ are parallel, and a polygon $B_1B_2 \dots B_{2n}$ inscribed in K with vertices that are just the points of contact of the circumscribed polygon with the curve (Diagram 100; the possibility that any adjacent vertices of the

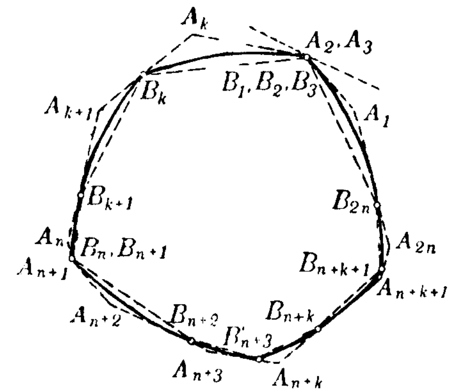


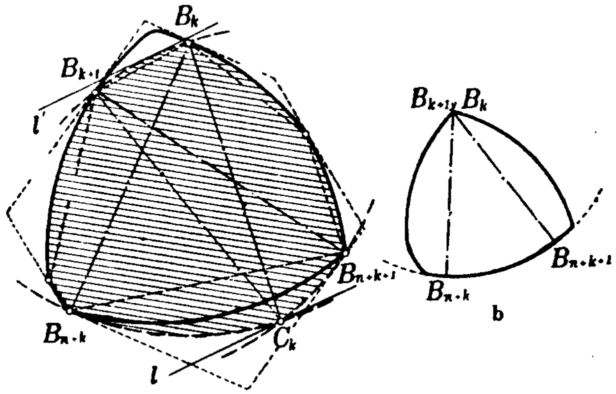
Diagram 100

circumscribed or of the inscribed polygon may coincide, say A_2 and A_3 or B_1, B_2, and B_3 in our drawing, is not to be excluded). We shall show the existence of a curve of constant width h consisting of circular arcs of radius h, which is inscribed in $A_1A_2 \dots A_{2n}$ and circumscribed about $B_1B_2 \dots B_{2n}$.

If B_k and B_{n+k} are two opposite vertices of the inscribed polygon, then the chord B_kB_{n+k} of the curve K is a diameter (since B_k and B_{n+k} are points of contact of K with a pair of parallel supporting lines). From this it follows that the length of the chord B_kB_{n+k} is h (Exercise 7-4); likewise the length of the chord $B_{k+1}B_{n+k+1}$ is h.

We now replace the opposite arcs B_kB_{k+1} and $B_{n+k}B_{n+k+1}$ of K by new circular arcs of radius h in such a way that the curve retains its constant width h and remains inscribed in $A_1A_2 \dots A_{2n}$. For this purpose we draw circles of radius h about the points B_k and B_{k+1}; let C_k denote the point of intersection of these circles which lies on the same side of the chord B_kB_{k+1} as the points B_{n+k}, B_{n+k+1} (Diagram 101 a). The point C_k is at distance h from the points B_k and B_{k+1}. Now we draw a circular arc of radius h about C_k which joins B_k and B_{k+1}. We replace the arc $B_{n+k}B_{n+k+1}$ of K by the circular arcs $B_{n+k}C_k$, and C_kB_{n+k+1} of radius h with centers at B_k, B_{k+1}, and the arc B_kB_{k+1} by a circular arc with center at C_k.

It is evident that the new curve K' obtained this way is again a curve of constant width h. In fact, if one of two parallel supporting lines l and l' of K' touches the arc $B_{n+k}C_k$, then the other passes through the vertex B_k; if l goes through C_k, then l' is tangent to the arc B_kB_{k+1}; if l touches the arc C_kB_{n+k+1}, then l' passes through the vertex B_{k+1}. Thus the curve K' remains inscribed in the polygon $A_1A_2 \ldots A_{2n}$ and circumscribed about the polygon $B_1B_2 \ldots B_{2n}$.†



a

Diagram 101

In case B_k and B_{k+1} coincide, the diameters B_kB_{n+k} and $B_{k+1}B_{n+k+1}$ intersect on the curve, so that the arc $B_{n+k}B_{n+k+1}$ is a circular arc of radius h about the point $B_k = B_{k+1}$ (Diagram 101 b; see Exercises 7-6 and 7-8). If we carry out the same construction for every pair of opposite arcs of K, we obtain a curve K_0 of constant width, consisting only of circular arcs of radius h, which is inscribed in the polygon $A_1A_2 \ldots A_{2n}$ and circumscribed about the polygon $B_1B_2 \ldots B_{2n}$ (Diagram 102).

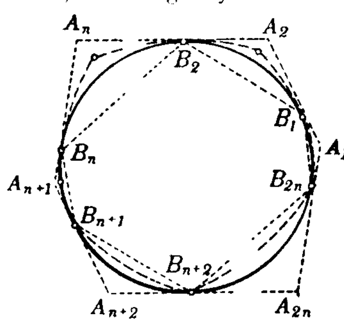
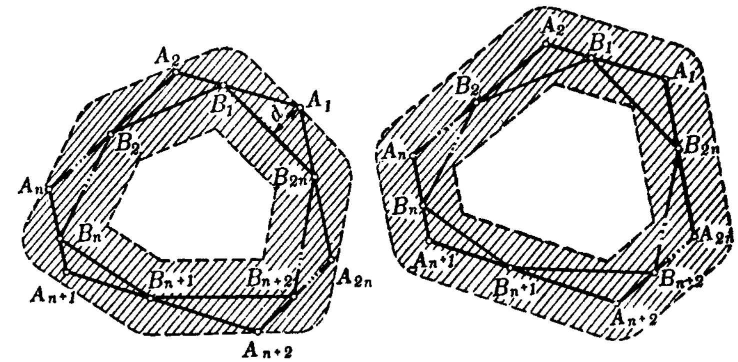


Diagram 102

It remains only to show that by proper choice of the polygon $A_1A_2 \ldots A_{2n}$ we can make the distance between K_0 and K as small as we please. It follows at once from the fact that K and K_0 are both inscribed in $A_1A_2 \ldots A_{2n}$ and circumscribed about $B_1B_2 \ldots B_{2n}$ that the distance between these curves is no greater than the

† In this construction, of course, we can interchange the roles of the points A_k, A_{k+1} and A_{n+k}, A_{n+k+1}; in this case we obtain another curve K'' which also has the desired property.

distance between the polygons; for if an r-neighborhood of $A_1A_2 \ldots A_{2n}$ encloses the polygon $B_1B_2 \ldots B_{2n}$ and the r-neighborhood of $B_1B_2 \ldots B_{2n}$ encloses the polygon $A_1A_2 \ldots A_{2n}$, then K cannot extend beyond the r-neighborhood of K_0, nor K_0 beyond the r-neighborhood of K. Thus if we prove that *the distance between* $A_1A_2 \ldots A_{2n}$ and $B_1B_2 \ldots B_{2n}$ *can be made as small as we wish*, it will follow that the distance between K_0 and K can also be made as small as we wish.



a Diagram 103 b

Moreover it is obvious that the distance between the polygons $A_1A_2 \ldots A_{2n}$ and $B_1B_2 \ldots B_{2n}$ is equal to the largest of the distance between the vertices $A_1, A_2, \ldots, A_{2n}$ and the corresponding sides $B_{2n}B_1, B_1B_2, \ldots, B_{2n-1}B_{2n}$ of $B_1B_2 \ldots B_{2n}$. In fact, if d is this distance, then $A_1A_2 \ldots A_{2n}$ lies entirely in the d-neighborhood of $B_1B_2 \ldots B_{2n}$ (Diagram 103 a) and conversely (Diagram 103 b). Now we assume that all angles of the polygon are equal, that is, all are equal to

$$\frac{(2n-2)\cdot 180^\circ}{2n} = \left(1 - \frac{1}{n}\right)\cdot 180^\circ$$

(the sum of the angles of a polygon of $2n$ sides equals $(2n-2)\cdot 180^\circ$). Consider the triangles $B_{2n}A_1B_1$, $B_1A_2B_2$, ..., $B_{2n-1}A_{2n}B_{2n}$. The base of each of these triangles is less than h (Exercise 7-3) and the angle at the vertices of the polygon is $(1 - 1/n)\,180^\circ$. Hence the altitude of each triangle is less than the altitude of a segment that includes an angle $(1 - 1/n)\,180^\circ$ above a chord of length h (Diagram 104).

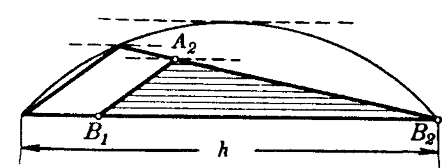


Diagram 104

However, as n increases, the height of this segment becomes as small as we please [it is easy to compute that it is equal to $(h/2)\tan(180^\circ/2n)$]. Hence, by choosing a sufficiently large number of sides for the polygon $A_1A_2 \dots A_{2n}$ with $2n$ sides, we can make the distance between $A_1A_2 \dots A_{2n}$ and $B_1B_2 \dots B_{2n}$, and hence between K_0 and K, as small as we wish.

From the theorem just proved it follows at once that for each curve K of constant width h we can always find a sequence K_1, K_2, ..., K_m, ..., of curves of constant width h, where each of the curves consists of circular arcs of radius h, such that K is the limit of the sequence. For this purpose it is only necessary to require that the distance between K_1 and K be less than 1, the distance between K_2 and K be less than $\frac{1}{2}$, ..., the distance between K_m and K be less than $1/m$, etc. This theorem (called the *Approximation Theorem*†) is often useful in studying properties of arbitrary curves of constant width. In particular, the most difficult of the exercises stated earlier in this section (see Exercises 7-11 and 7-12) can be solved by using this theorem (see Exercises 7-19 and 7-20).

7-19. Obtain a new proof of Barbier's Theorem from the Approximation Theorem. (See Exercise 7-11.)

7-20.** From the Approximation Theorem, obtain a new proof that of all figures of constant width h the Reuleaux triangle bounds the smallest area.

A convex body in three-dimensional space is called a *body of constant width h* if the distance between any pair of parallel supporting planes is h. There are infinitely many bodies of constant width besides the sphere. However, examples of such bodies are more complicated than examples of plane curves of constant width, since there are no nonspherical bodies of constant width whose surfaces are entirely composed of portions of spheres. The simplest example of a nonspherical body of constant width is the body obtained by rotation of a Reuleaux triangle about its axis of symmetry (Diagram 105).

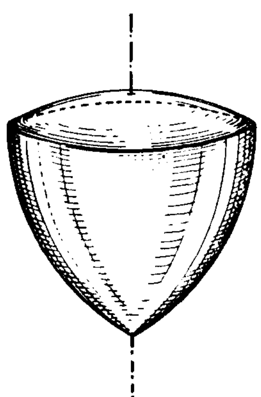


Diagram 105

As examples of bodies of constant width which are not produced by rotation there are the "tetrahedra of constant width." These are

† From the Latin *approximare*, meaning approach. Theorems of this kind, which permit the representation of complicated mathematical objects as limits of sequences of simpler objects, play a very important role in contemporary mathematics.

bodies which can be regarded as spatial analogues of the Reuleaux triangle. There are two different types of "tetrahedra of constant width." They are constructed as follows. Describe a sphere of radius h about each vertex of a regular tetrahedron of side h. The intersection of these four spheres forms the body shown in Diagram 106 a. Next we remove

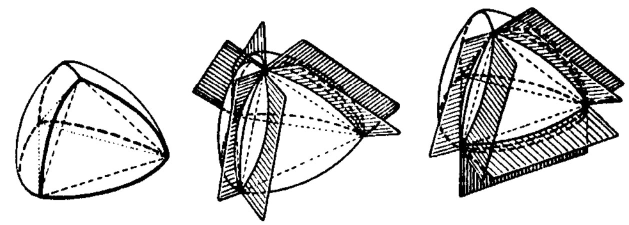

a b c

Diagram 106

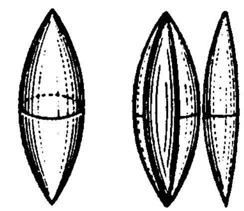

d e

Diagram 106


a b

Diagram 107

the parts of the surface of this body which lie inside an angle between two surfaces, either the parts vertical to three tetrahedral edges which meet in a vertex (Diagram 106 b), or to three edges which bound a face (Diagram 106 c). These parts of the surface of the body are replaced by three pieces of surface generated by the rotation of the circular arcs bounding the given surface about the corresponding edges, that is, by portions of the surface of the spindle-shaped bodies shown in Diagram 106 d; a section of such a portion of surface is shown in Diagram 106 e. The two different types of bodies thus obtained are shown in Diagrams 107 a and 107 b. It is easy to verify that both types have constant width h. The two "tetrahedra of constant width" have the same surface and the same volume. We shall assume without proof that the "tetrahedra of constant width h" have the least volume among all bodies of constant width h.

Bodies of constant width h may have different surface areas; in other words, in space there is no analogue of Barbier's Theorem. However bodies of constant width in space have, instead, another remarkable property.

In three-dimensional space, the *projection* of a convex body P is the plane figure obtained by a parallel projection of P on some plane (Diagram 108). Since all projections of a body P of constant width h obviously produce plane figures of constant width h, it follows from Barbier's Theorem that all projections of P have exactly the same

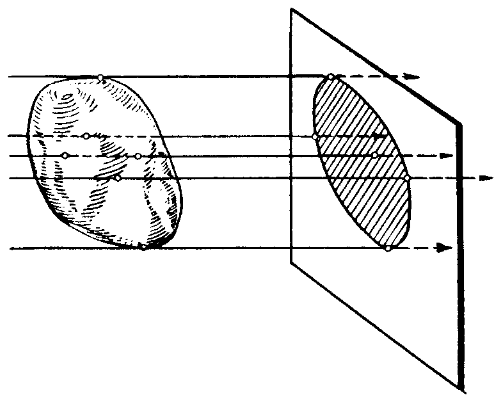


Diagram 108

perimeter πh (the length of the curve bounding the projection). A convex body is said to have a *constant perimeter* if all of its projections have the same perimeter.

Thus we see that *all bodies of constant width are also bodies of constant perimeter*. The converse also holds: *Every body of constant perimeter is a body of constant width*; however the proof of this theorem is complicated.†

† See T. Bonnesen and W. Fenchel: *Theorie der konvexen Körper* (Theory of convex bodies). Ergebnisse der Mathematik und ihrer Grenzgebiete 3, Berlin, 1934.

8

CURVES WHICH CAN BE ROTATED IN AN EQUILATERAL TRIANGLE (Δ-CURVES) AND RELATED CURVES

The contents of this section are closely related to those of the preceding one, which was devoted to curves of constant width. Hence this section should not be read before Section 7.

Knowledge of Section 1 is sufficient for the solution of the first eleven exercises and for Exercises 8-18 through 8-20. To solve Exercise 8-12, the theorem of Exercise 3-4 b is essential; if this section is read *before* Section 3, then it is necessary to examine Exercise 3-4 b carefully before attempting Exercise 8-12 (one need merely read through the solution). Exercises 8-13, 8-14, and 8-21 assume knowledge of Exercises 4-1 through 4-11. To understand the parts of this section which are in fine print (including Exercise 8-17), knowledge of all of Section 4 is necessary.

Curves of constant width (see Section 7) may be defined as convex curves that can be rotated in a square so that they always touch all sides of the square (see page 72, Diagram 93 b). It is natural to pose questions about the properties of convex curves that can be rotated freely inside other polygons. In this section we consider curves that can be rotated freely in the interior of an equilateral triangle so that they always remain in contact with all sides of the triangle (Diagram 109 a) or, what is the same thing, we consider curves all of whose circumscribed

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Dedicated to the memory of an outstanding scholar

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CONVEX FIGURES

I. M. Yaglom and V. G. Boltyanskiĭ

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Translated by

PAUL J. KELLY and LEWIS F. WALTON

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