Theorem 5. Let F be a k-dimensional flat and S an open convex set in E^n such that $F \cap S = \emptyset$. Then for $0 \leq k \leq n - 2$, there exists a $(k + 1)$-dimensional flat F^* such that $F^* \supset F$ and $F^* \cap S = \emptyset$.

Corollary 1. Let F be a k-dimensional flat $(0 \leq k < n)$ and S an open convex set in E^n. If $F \cap S = \emptyset$, then there exists a hyperplane H such that $H \supset F$ and $H \cap S = \emptyset$.

Theorem 6. Suppose A and B are convex subsets of E^n. If int $A = \emptyset$ and $B \cap \text{int } A = \emptyset$, then there exists a hyperplane separating A and B.

Theorem 7. Suppose A and B are convex subsets of E^n such that $\dim(A \cup B) = n$. Then A and B can be separated by a hyperplane if and only if relint $A \cap$ relint $B = \emptyset$.

Definition 9. Let S be a convex subset of E^n. A point x in S is an extreme point if it does not belong to the relative interior of any line segment contained in S. Equivalently, x is an extreme point if $y, z \in S$, $0 < \lambda < 1$, and $x = \lambda y + (1 - \lambda)z$ imply $x = y = z$. The set of all extreme points of S is the profile of S.

EXAMPLES. The profile of a polygon in E^2 is the set of vertices. The profile of a closed ball is its boundary. The profile of an open set (or a closed half-space) is empty. Clearly, if x is in the profile of a convex subset S of E^n, then $x \notin \text{conv}(S \sim \{x\})$.

Theorem 8. A compact convex subset of E^n is the convex hull of its profile.

IV. Theorems of Helly's and Kirchberger's Types

Theorem 1 (J. Radon). Let $S \equiv \{x_1, x_2, \ldots, x_r\}$ be any finite set of points in E^n. Then for $r \geq n + 2$, S can be partitioned into two disjoint subsets S_1 and S_2 such that conv $S_1 \cap$ conv $S_2 \neq \emptyset$.

Radon used Theorem 1 to prove the following theorem.

Theorem 2 (E. Helly). Let $\mathcal{F} \equiv \{B_1, B_2, \ldots, B_r\}$ be a family of r convex sets in E^n with $r \geq n + 1$. If every subfamily of $n + 1$ sets in $\mathcal{F}$ has a nonempty intersection, then $\cap_{i=1}^{r} B_i \neq \emptyset$.

Theorem 2 introduces a class of theorems of the following form: Let $\mathcal{F}$ be a collection of sets, and k a fixed positive integer. If every k sets in $\mathcal{F}$ have property $\mathcal{P}_1$, then the collection $\mathcal{F}$ has property $\mathcal{P}_2$.

Theorem 3 (P. Kirchberger). Let P and Q be nonempty compact subsets of E^n. Then P and Q can be strictly separated by a hyperplane if and only if for each set T consisting of $n + 2$ or fewer points of $P \cup Q$ there exists a hyperplane that strictly separates $T \cap P$ and $T \cap Q$. Theorem 3 is equivalent to the following formulation.

Suppose P and Q are nonempty compact subsets of E^n. Then there exists a closed half-space S such that $S \supset P$ and $S \cap Q = \emptyset$ if and only if for each set T consisting of $n + 2$ or fewer points of $P \cup Q$ there exists a closed half-space S_T such that $S_T \supset (T \cap P)$ and $S_T \cap (T \cap Q) = \emptyset$.

The following question immediately arises from this formulation: If S is replaced by some other figure (such as sphere, cyclinder, or parallelotope), what simple condition will ensure the existence of such a figure containing P and disjoint from Q?

V. Some Topics in E^2

This section is mainly concerned with some special topics on convex sets in E^2. Some of these results have been generalized to E^n for $n \geq 3$.

Definition 1. A set S is a convex body if it is compact, convex, and has a nonempty interior. A set is a closed convex surface if it is the boundary bd S of a convex body S. If S is planar, then it is a plane convex body, and its boundary bd S is a closed convex curve.

The term "convex" in the name "closed convex surface (respectively, curve)" is used in association with the set to which the surface (respectively, curve) is a boundary. In fact we can prove that a closed convex surface or curve is never itself a convex set. Also, the term "closed" in these names might seem to be redundant since boundaries are automatically closed sets. Initially, however, the term "closed" in the name "closed convex surface (respectively, curve)" was introduced to indicate that such a surface (respectively, curve) actually encloses a region in the sense that the sphere $S^3(x, \delta)$ (respectively, the circle $S^2(x, \delta)$) encloses the open ball $B^3(x, \delta)$ (respectively, the open disk $B^2(x, \delta)$), where $S^n(x, \delta) \equiv \{y \in E^n \mid d(y, x) = \delta\}$, $B^n(x, \delta) = \{y \in E^n \mid d(y, x) < \delta\}$.

Definition 2. Let S be a nonempty compact subset of E^2, and let l be a line (or line segment) in E^2. The width of S in the direction of l is the distance between the two parallel lines of sup-

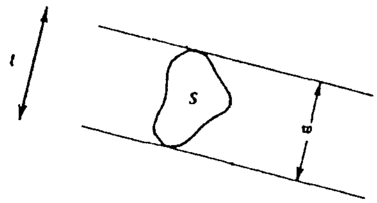

FIG. 4. Width of set S.

port to S, which are perpendicular to l and contain S between them (See Fig. 4.)

Definition 3. Let S be a nonempty bounded subset of E^2. The diameter of S is the number

$$d \equiv \sup_{\substack{x \in S \\ y \in S}} \|x - y\|.$$

Theorem 1. The diameter of nonempty compact set S in E^2 is equal to the maximum width of S.

Definition 4. In general, a set in E^2 has different widths in different directions. If the width of a set in E^2 is the same for all directions, then the set is said to be of constant width.

EXAMPLE 1. The simplest example of a set of constant width is a circle. Noncircular sets of constant width were first studied by Euler in the 1770s, and since then by many other mathematicians. Among the early contributors was F. Reuleaux, for whom the following Reuleaux triangle is named: Let ΔABC be an equilateral triangle with the length of each side equal to w. With each vertex as center, draw the smaller arc of radius w joining the other two vertices. The union of these arcs is the boundary of a Reuleaux triangle. Clearly it is a noncircular set of constant width w. Indeed, given any two parallel supporting lines, one of them will pass through a vertex of ΔABC, while the other will be tangent to the opposite arc. Thus the distance between them is w. (See Fig. 5.)

Definition 5. Let S be a compact subset of E^2. A point $x \in \text{bd } S$ is a corner point (or vertex)

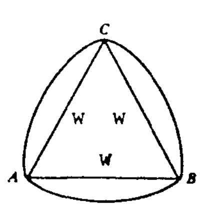

FIG. 5. Reuleaux triangle.

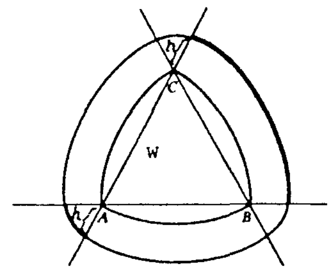

FIG. 6. Realeaux triangle added to a circle.

if there exists more than one line of support to S at x.

EXAMPLE 2. While a Reuleaux triangle has corner points, other noncircular sets of constant width do not. For example, if R is a Reuleaux triangle of width w and U is a circle of radius h centered at the origin, then $R + U$ is a set of constant width $w + 2h$ that has no corners. Another way to construct $R + U$ is by extending each side of the ΔABC a distance h in both directions. Then $R + U$ is formed by the six circular arcs each two of which have center at each vertex of the ΔABC and radii $w + h$ and h, respectively, as indicated by the thicker arcs for the vertex A (see Fig. 6).

Theorem 2. Let S be a plane convex body with constant width w. If $x \notin S$, then the diameter of $S \cup \{x\}$ is greater than w, and the converse is also true. In other words, plane convex bodies are complete in the sense that no point can be added to them without increasing their diameter.

Definition 6. A circle of largest diameter which lies entirely in a plane convex body S is an incircle of S, and the circle of smallest diameter that encloses S is the circumcircle.

Theorem 3. The incircle and the circumcircle of a plane convex body with constant width w are concentric, and the sum of their radii is equal to w.

Definition 7. The perimeter of a plane convex body S is the length of its boundary bd S, which is equal to the supremum of the perimeters of all the inscribed convex polygons of S. The area of S is equal to the supremum of the areas of all the inscribed convex polygons of S.

In Definition 7 we could have used the infimum of circumscribed polygons about S instead of the supremum of inscribed polygons. In both cases, as the number of sides increases and the side length decreases, the area and the perimeter

of the polygons appro
body S.

For plane convex bo
ing remarkable theorem

Theorem 4 (E. Barb
curves of constant widt
eter πw.

However, for the ar
constant width in E^2
theorem.

Theorem 5. Among
constant width w in E^2,
has the greatest area a
of width w has the lea

Applications. **1.** S
width can be turned in
maintains contact with
a drill shaped like a
used to bore a hole wit
ter of the triangle mov
and the corners of the
edges.

2. To project a m
make a brief quick mo
closed) followed by a
the shutter open). The
vides this intermittent
Reuleaux triangle.

Let P be a plate that
tally but not vertically (
gular hole in the ce
Reuleaux triangle R o
rotated continuously a
move back and forth in
R rotates through 120° f
will move to the right.
next 60°, P will remain
turn, P will move back
by a pause while R fi
complete revolution. T
of R is changed into th
tion of P.

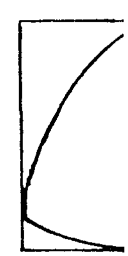

FIG. 7. Set with constan
of a square.

of the polygons approach that of the convex body S.

For plane convex bodies we have the following remarkable theorem.

Theorem 4 (E. Barbier). All closed convex curves of constant width w have the same perimeter πw.

However, for the area of a compact set of constant width in E^2 we have the following theorem.

Theorem 5. Among all the compact sets of constant width w in E^2, the circle of diameter w has the greatest area and the Reuleaux triangle of width w has the least.

Applications. **1.** Since a set of constant width can be turned inside a square so that it maintains contact with all four sides (see Fig. 7), a drill shaped like a Reuleaux triangle can be used to bore a hole with straight sides. The center of the triangle moves in an eccentric path, and the corners of the triangle are the cutting edges.

2. To project a movie film, the film must make a brief quick movement (with the shutter closed) followed by a momentary pause (with the shutter open). The gear mechanism that provides this intermittent motion is based on the Reuleaux triangle.

Let P be a plate that is free to move horizontally but not vertically (see Fig. 8). Cut a rectangular hole in the center of P and place a Reuleaux triangle R of width w in it. If R is rotated continuously about one corner c, P will move back and forth intermittently. In Fig. 8 as R rotates through 120° from position (a) to (b), P will move to the right. As R rotates through the next 60°, P will remain still. During the next 120° turn, P will move back to the left, again followed by a pause while R finishes the last 60° of its complete revolution. Thus the continual rotation of R is changed into the intermittent linear motion of P.

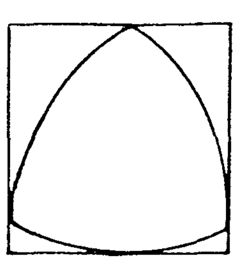

FIG. 7. Set with constant width can be turned inside of a square.

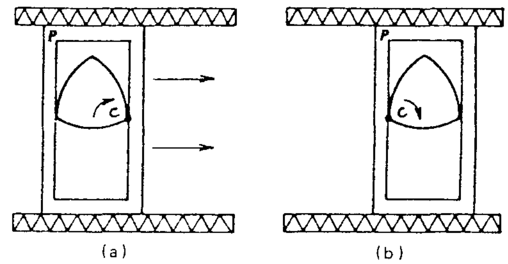

FIG. 8. Use of Reuleaux triangle in a film projector.

Related to the concept of the diameter of a set is that of a universal cover.

Definition 8. A compact subset K of E^2 is a universal cover in E^2 if any subset S of E^2 having diameter 1 can be covered by a congruent copy of K.

Since the diameter of a set S is the same as the diameter of cl(conv S), we need only consider compact convex sets S when verifying that K is a universal cover. Mathematicians have been interested for many years in finding the smallest universal cover having a given shape. The solution to the problem for squares and circles (disks) can be obtained as an application of a Helly-type theorem.

Theorem 6. The smallest square (respectively, hexagon, equilateral triangle) that is a universal cover in E^2 has sides of length one (respectively, $1/\sqrt{3}$, $\sqrt{3}$).

Theorem 7. The smallest circle that is a universal cover in E^2 has radius $1/\sqrt{3}$.

From Theorem 6 we are led to the following general question: For each fixed positive integer k, what is the smallest regular k-gon that is a universal cover in E^2? Theorem 6 shows that for $k = 3, 4, 6$ the answer is the k-gon circumscribed about a circle of diameter 1. The answer for k other than these is not known. It is still not true for $k > 6$ if we conjecture that any regular k-gon circumscribed about a circle of diameter 1 would be a universal cover in E^2. Indeed, as k increases, the circumscribed k-gons approach the circle, but the smallest circular universal cover was found to have a diameter $2/\sqrt{3}$.

The most famous problem about universal covers is the following Lebesque's problem: What is the minimum area A of a universal cover in E^2? We know that A satisfies the inequality

$$0.785 \approx \frac{\pi}{4} \leq A \leq \frac{\sqrt{3}}{2} \approx 0.886,$$

but the exact value of A is not known.

Volume 3 Co–Cryp

ENCYCLOPEDIA OF PHYSICAL SCIENCE AND TECHNOLOGY

An Encyclopedic Reference Complete in Fifteen Volumes, with Volume 15 the Index Volume

ROBERT A. MEYERS, EDITOR
TRW, INC.

ACADEMIC PRESS, INC.
Harcourt Brace Jovanovich, Publishers
Orlando San Diego New York Austin
Boston London Sydney Tokyo Toronto