FIBONACCI NUMBERS

Graphic and Visual Communication

A Magic Ratio Recurs Throughout Art and Nature

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The following article is reprinted from *Smithsonian*, December 1975, as corrected February 1976. It makes an interesting connection between art and nature and mathematics. The ratio it describes thus connects that which constitutes effective visual communication and that which can be described by a numeric constant.

His neighbors in Pisa called him *Bigollone*, "the Blockhead." His real name was Leonardo and he was the son of Bonaccio, a customs official whose name meant "Simpleton." Back in the 13th century, when discoverers and inventors rated slightly below sorcerers on the social scale, Leonardo the Blockhead stumbled onto one of the great mysteries of the universe.

History speaks of him as Fibonacci, which literally means "Son of the Simpleton." He grew up in the North African city of Bugia, where his father was stationed. The inquisitive youngster was educated by Muslims of the Barbary Coast, who taught him the Arabic system of numerals. He quickly realized that "98" was much simpler to work with than "XCVIII." Returning home to Pisa, the young scholar apparently let his mind wander as his feet shuffled through town, and when inspiration hit he would grab a bit of chalk from his pocket and absent-mindedly scrawl numbers onto the nearest wall.

In 1202, at the age of 27, Fibonacci published *Liber Abaci* (The Book of the Abacus), the historic manuscript that was the principal means of introducing Arabic numerals to the European world. This accomplishment alone was a spectacular contribution to scientific thought, but a small section of the book contained a theoretical problem that has fascinated scholars for centuries. Fibonacci's solution to the problem resulted in a mysterious series of numbers that have led mathematical sleuths on a merry chase through the realms of art, architecture, oceanography, botany, biology, astronomy and music. Yet, to this day, no one can fully explain their significance.

The problem, as posed by Fibonacci, is this: "Someone placed a pair of rabbits in a certain place, enclosed on all sides by a wall, to find out how many pairs of rabbits will be born there in the course of one year, it being assumed that every month a pair of rabbits produces another pair, and that rabbits begin to bear young two months after their own birth."

There is no record that Fibonacci actually bred any rabbits. But his own fertile mind produced a succession of numbers that grew dramatically.

The mathematician reasoned that in the first month the original pair did not breed, but thereafter produced an additional pair each month. By the fourth month the first offspring reached maturity and began breeding also. Each month a new group of 2-month old animals would breed, multiplying the rabbit population at an ever-increasing speed. At the end of 12 months the original rabbits and their offspring would number 233 pairs!

Fibonacci listed the total pairs of rabbits at the end of each month, and created the following sequence of numbers . . . 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233.* Upon analyzing the figures, he realized that each number was the sum of the two preceding numbers. In other words, 1 plus 2 equals 3; 2 plus 3 equals 5; 3 plus 5 equals 8; 5 plus 8 equals 13, and so on, *ad infinitum*. Though the rabbits cannot continue multiplying indefinitely (age will take its toll), the Fibonacci sequence apparently goes on forever – the last two numbers in the series always combining to produce yet another.

The 100th number in the sequence is a whopping 354,224,848,179,261,915,075!

The series may appear to be composed of nothing more than random numbers; in fact they are as precise as nature itself. Students of the Fibonacci sequence discovered that each number bears a special relationship to the numbers surrounding it. If you divide a Fibonacci number by the next highest number, you will discover that it is precisely 0.618034 times as large as the number that follows. (The precise figure appears when the Fibonacci

*If the number of rabbits before the beginning of the first month (0) and at the beginning of the first month (1) are included, the Fibonacci series begins 0, 1, 1, 2, 3, 5, 8, 13, 21 ....

numbers are large enough to be precise; it works for all numbers after the 14th in the sequence.)*

And 0.618034 is a magic number.

The golden proportion of 0.618034 to 1 is the mathematical basis for the shape of playing cards and the Parthenon, sunflowers and snail shells, Greek vases and the great spiral galaxies of outer space.

The Greeks based much of their art and architecture on the proportion of 0.618034 to 1. They called it "the golden mean." The Greeks apparently did not understand the mathematical basis for the golden mean, but they knew it pleased the eye. They defined the golden mean as the point that divided a line into two parts in such a manner that the smaller part was in proportion to the larger part as the larger part was to the entire line. They determined that point by various geometric exercises.

Johann Kepler, the astronomer and no mean mathematician himself, considered it an outstanding achievement. He said: "Geometry has two great treasures: one is the theorem of Pythagoras; the other, the division of a line into extreme and mean ratio [the golden mean]. The first we may compare to a measure of gold; the second we may name a precious jewel."

Perhaps the earliest evidence of Man's knowledge of the golden mean is still visible today on the plateau of Giza in Egypt. Some scholars say that the Egyptians based their work on a loose proportion of 5 to 8, two early numbers of the Fibonacci sequence that approximate the golden mean, but are slightly off. The average ratio of altitude to base of all the pyramids at Giza is about 0.625 or 5 to 8. There is scholarly disagreement as to the exactness of Egyptian knowledge, confounded by the fact that age has crumbled the capstones of the pyramids so that their precise original heights must be estimated. At least one serious scholar claims that the Great Pyramid was originally 484 feet, 5 inches in height. Reduced to inches – there is reason to believe the Egyptians worked in inches – the height would then be 5,813 inches (5, 8, and 13 are numbers in the Fibonacci sequence). If that figure is correct, then the pyramid represents an astounding calculation. If the height of the pyramid was considered to be the radius of a circle, the circumference would be 36,524.2 inches. Did the Egyptians know that the exact length of a year is 365.242 days?

The Egyptians were obviously appreciative of the triangle. From it they developed the pentagon and its star-shaped sister the pentagram, both of which are inscribed in a circle by drawing a series of triangles. Both forms are remarkable displays of the golden mean, with intricate divisions of the figures separating lines into proportions of 0.618034 to 1.

The mystery deepens when the golden proportion is expanded to a four-sided figure. If you draw a rectangle so that the smaller side is 0.618034 as long as the larger side, you have created a unique and pleasing piece of art. Both Pythagoras and Euclid called it the rectangle of the Divine Section. In 1876 German psychologist Gustav Theodor Fechner measured the dimensions of thousands of common rectangles, such as playing cards, windows, writing pads and book covers. He found that, on the average, their proportions were close to the golden mean. Without knowing why, the designers had realized that the golden rectangle was a pleasant shape. Fechner and a successor, Wilhelm Max Wundt, tested hundreds of individuals to determine their preferences for rectangles of various proportions. About 75 percent preferred the proportions of the golden mean.

The golden mean, triangle, and rectangle all formed the basis for much of the classical art and architecture of the Greeks. Plato wrote:

> It is impossible to join two things in a beautiful manner without a third being present, for a bond must exist to unite them, and this [bond] is best achieved by a proportion.
>
> For, if of three magnitudes the mean is to the least as the greatest to the mean, and, conversely, the least is to the mean as the mean to the greatest – then is the last the first and the mean, and the mean the first and the last. Thus are all by necessity the same, and since they are the same, they are but one.

The front of the Parthenon, including the pediment when it was intact, would have fitted almost exactly into a golden rectangle. The beautiful Greek vases are designed to please the eye with intricate and exquisite golden proportions. Greek figure sculptors chose the navel as the golden mean of the body, with the upper area again being divided into golden proportions at the neck and eyes. The Greek sculptor Phidias was especially fond of golden ratios and his work exhibits them in such proportions as the relation of the width of the throat to the width of the head; the narrow part of the thigh to the widest part; the width of the forearm to the wrist; and so on, with beautiful attention to proportional detail.

The secret was lost with the fall of Greece, but it began to surface in the 16th century. Another Leonardo, named da Vinci, became fascinated with "geometrical recreations" and utilized them in painting and sculpture. In 1509 he collaborated with a monk named Luca Pacioli and produced a book which they entitled *De Divina Proportione*.

Many of the masters began to proportion their canvases according to the golden mean, from Piero della Francesca and Bellini to Poussin and, in the 19th century, Seurat (the latter was said to "attack every canvas by the golden section").

In the early 20th century, Yale scholar Jay Hambridge defined two types of geometric symmetry in classical and modern art. One he defined as static symmetry based on the straight lines and sharp angles seen in the lifeless

*A *Smithsonian* reader pointed out that the ratio is not precisely 0.618034. It is equal to $(\sqrt{5} - 1)/2$, or 0.6180339887 ... and so on.

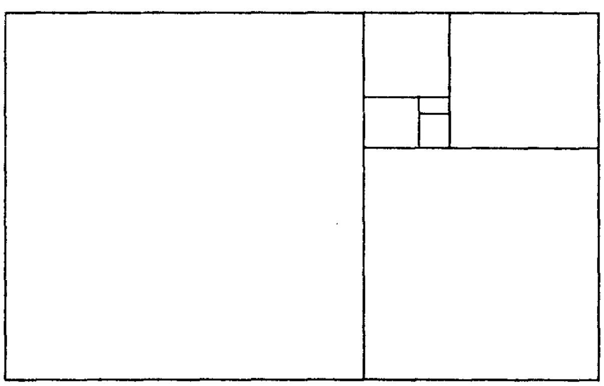
forms of Roman art. The more imaginative form he called dynamic symmetry, which was based on what he called "whirling squares." This is the vibrant, alive, moving style of art, Hambridge said, which was more characteristic of the Greeks. An analysis of "whirling squares" brings us a step closer to understanding the esthetic appeal of the golden mean.

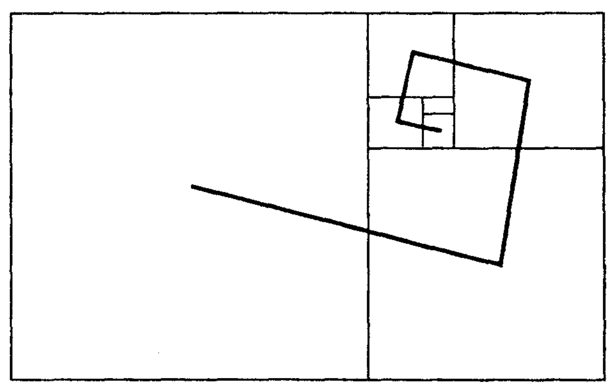
It was Jakob Bernoulli in the 17th century who began to suspect that the golden mean was intimately connected with the strange twists of nature. To see the connection, we must look inside the golden rectangle.

If you draw a square into one end of a golden rectangle, you are left with another, smaller golden rectangle (Figure 1). This division can be performed again and again until you are left with a succession of Hambridge's whirling squares which can be reduced or expanded *ad infinitum*. When you connect the center points of the successive squares you draw a sort of stiff spiral that works its way ever larger. Any section of the spiral is 0.618034 as large as the remainder of the spiral.

But golden proportions do not limit themselves to straight lines. When you curve out the lines, you have drawn a golden spiral. Bernoulli named it the logarithmic spiral, and he noted that any line drawn from the center of the spiral will intersect it at precisely the same angle as any other line. For this reason it is also known as the equiangular spiral. Bernoulli was so impressed with it that he ordered it engraved on his tombstone. Much as Christopher Columbus discovered America after Leif Ericson (not to mention the Indians), much as Fibonacci discovered the proportions of the golden mean after the Greeks and the Egyptians, Bernoulli discovered the logarithmic spiral long after nature herself. "Nature uses this as one of her most indispensable measuring rods, absolutely reliable, yet never without variety, producing perfect stability of purpose without the slightest risk of monotony," wrote C. Arthur Coan. "... We shall find it flung broadcast throughout all nature."

Legend has it that the mollusk originated from the earthworm. One day the worm, rendered frisky by the sun, emancipated itself, brandished its tail and twisted itself into a corkscrew for the sheer fun of it. If that is the case, there were frisky worms in abundance who twisted themselves into a wise and pleasing variety of logarithmic spirals. Shellfish from among the earliest forms of life exhibit the spiral. "It is possible to trace even back in the plankton phase, distinct expressions of spiral organization associated with Fibonacci ratios," write A. H. Church. The globigerinae, planorbic planorbisvortex, terebra, turritellae and trochida all build their tiny bodies according to the logarithmic spiral, as does the snail.





Above is a golden rectangle: the depth is 0.618034 of the width. When the left side was squared, another smaller golden rectangle was created on the right side. Then the square at bottom right created the rectangle at top right, and so on. In the figure at upper right, the centers of the squares thus formed have been connected. When the connecting line is smoothed into a curve, right, you have the golden spiral so rampant in nature.

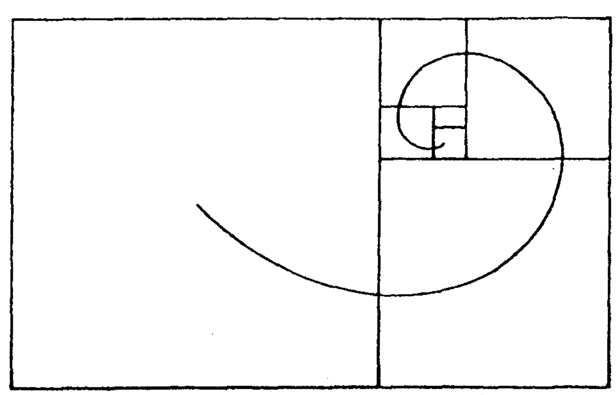


Figure 1. Drawing a Golden Spiral

One of the most spectacular sea spirals is created by the chambered nautilus. As the animal grows it builds ever larger compartments in an expanding spiral formation. When each new apartment is ready for occupancy, the animal crawls forward, slamming the door behind it with a layer of mother-of-pearl. The old living quarters remain filled with gas and air, so that the whole structure remains buoyant in spite of its massive build. Not only does the nautilus build the outside of its shell in a spectacular spiral, but the inner partitions are also curved in the same graceful form.

Surfers shooting the gap underneath a breaking wave are covered by a logarithmic spiral of ocean spray. The shoreline of Cape Cod is a logarithmic spiral. The spiral formation is so closely allied with the relentless pounding of the sea that the great Swedish treatise on shipbuilding, *Architectura Navalis Mercatoria*, suggests that the most effective shape for the arms of an anchor is the same spiral.

Botany is a veritable gold mine of spirals and Fibonacci numbers. Many types of composite flowers exhibit striking examples. The outcome of "She loves me, she loves me not" is quite likely to depend upon a Fibonacci number of daisy petals. One researcher has discovered daisies with 21, 34, 55 and 89 petals. If the number of petals is not an exact Fibonacci number, it is likely to be close.

Sunflower seeds are arranged in logarithmic spirals flowing out from the center in both directions. If you count the number of seeds along the clockwise and counterclockwise spirals, you will be awarded with successive numbers in the Fibonacci sequence. Most sunflower heads seem to have spirals of 34 and 55 seeds. One giant sunflower grown in the Soviet Union reportedly exhibited spirals containing 89 and 144 seeds.

The scaly plates of the pineapple exhibit three distinct logarithmic spirals. Five wind sharply up in one direction, while 13 form a more gentle logarithmic spiral. In the other direction flows a third spiral comprised of eight plates.

Pine cones produce highly visible logarithmic spirals based on Fibonacci numbers. Pine needles tend to grow in clusters of 2, 3, or 5, depending upon the species.

None of these examples is always found to be perfect in the field. The Fibonacci sequence in nature is not a universal law. Rather, it seems to be a fascinating natural tendency that appears far too often to be discounted as mere chance.

A botanical phenomenon known as phyllotaxis produces even stranger evidence of Fibonacci numbers. Phyllotaxis refers to the arrangement of leaves upon the stalk of a plant. If you examine the stalk of almost any green plant, you can observe the Fibonacci numbers in action. Start at the bottom with a given leaf. Then move up the stalk, counting the leaves until you reach a leaf that is directly above the first one (do not count the first one). You will have a Fibonacci number. In addition, the number of times you circled the stalk (whether clockwise or counterclockwise) will be another Fibonacci number! For example, there are generally 13 buds arranged between two vertical lines on a pussy willow stem. To count the buds, you must circle the stalk five times.

Most of nature's horns, claws, and teeth seem to exhibit the spiral – perhaps most spectacularly in a ram's horn or a parrot's beak, but no less beautifully in an elephant tusk or a lion's claw. The extinct saber-toothed tiger and mammoth prove that the spiral has been twirling mysteriously for millions of years.

Sometimes the curve is barely perceptable, but is a portion of the logarithmic spiral just the same. Any dentist knows that he must pull a tooth in the direction of its curve.

Swedish astronomer Carl-Gustav Danver has studied the characteristic shape of the great galaxies of outer space. He developed techniques for photographing them and then manipulating the photographs so that the galaxies could be viewed as though on a flat plane. The predominant appearance, he reports, is that of arms of stars twirling outward in spectacular logarithmic spirals.

The continual occurrence of Fibonacci numbers and the golden spiral in nature explains precisely why the proportion of 0.618034 to 1 is so pleasing in art. Hambridge had said that Greek art was based on dynamic symmetry – whirling squares that seemed to vibrate with intense energy. Greek art seemed to be alive, because it was based on one of the most common proportions of life itself. This is why a golden rectangle is a pleasant shape. Man can see the image of life in art that is based on the golden mean.

Fibonacci's abracadabric rabbits pop up in the most unexpected places. The numbers are unquestionably part of a mystical natural harmony that feels good, looks good, and even sounds good. Music, for example, is based on the 8-note octave. On the piano this is represented by 8 white keys, 5 black ones – 13 in all. It is no accident that the musical chord that seems to give the ear its greatest satisfaction is the major sixth. The note E vibrates at a ratio of 0.62500 to the note C. A mere 0.006966 away from the exact golden mean, the proportions of the major sixth set off good vibrations in the cochlea of the inner ear – an organ that just happens to be shaped in a logarithmic spiral.

Occurrences of the Fibonacci sequence – like the sequence itself – never seem to end. Science can document their existence, but cannot fully explain them.

- Why, for example, does the epeira spider always spin its web into a logarithmic spiral?
- And why does a meteorite, when it ruptures the surface of the Earth, cause a depression that corresponds to a segment of the logarithmic spiral?
- Why do bacteria grow at an increasing rate that may be plotted along a logarithmic spiral?

And why does the tail of a comet curve gracefully away from the sun in a gigantic, blazing, logarithmic spiral? Applications of the Fibonacci sequence in science and art are so seemingly endless that an entire cult of mathematicians has been organized for further study of the phenomenon. Known as the Fibonacci Association, the group has published its own scholarly journal, *The Fibonacci Quarterly,* since 1963.

The coeditor of *The Fibonacci Quarterly* and a founder of the association is Verner E. Hoggatt Jr., a professor of mathematics at San Jose State College. His book, *Fibonacci and Lucas Numbers,* is a good source for anyone who wants to study the subject further.*

The scientific explanations have varied all the way from Sir James Jeans' "God is a mathematician" to Scrooge's "Bah, humbug!" Unquestionably, though, the Fibonacci sequence and the golden spiral are part of some recurring growth pattern.

The spiral tends to appear most frequently in nonliving matter, or in the nonliving tissues of living organisms, such as shells, horns and teeth. Britain's respected naturalist Sir D'Arcy W. Thompson pointed out that "the shell, like the creature within it, grows in size but does not change its shape; and the existence of this constant relativity of growth, or constant similarity of form, is of the essence." In other words, the very shape of the logarithmic spiral enables growth to occur without change in form. Fibonacci's theoretical rabbits increased in number but remained as rabbits. The chambered nautilus grows larger but continues to be no more nor less a chambered nautilus.

The great golden spiral seems to be nature's way of building quantity without sacrificing quality.

*So far as can be determined, this book is out of print. – *Ed.*

New PCS Magazine?

PCS is considering the feasibility of publishing a magazine of general interest to both engineers and communicators, to both PCS members and nonmembers. Such a magazine would not necessarily be intended to be archival; therefore, the format and topics covered would be entirely at our discretion, depending on the current needs of the readership constituency, as we perceive them.

At present, it is expected that the magazine would not supplant the *Transactions.* However, the feasibility study will take into account the entire body of PCS publications, including *Transactions* and *Newsletter.*

The magazine project is currently in the discussion stage by the PCS Administrative Committee. We would appreciate your expression of interest in such a publication and any thoughts you would like to contribute if this idea can be implemented. Your thoughts and ideas may be addressed to the Society at the address below:

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