Paradoxes of Probability

by Robert T. Kurosaka

Nonintuitive probabilities are all around us, leading to a proliferation of so-called sucker bets. In this column, I’ll analyze some of the more popular ones and simulate a couple of them with BASIC computer programs.

How about a small, friendly bet? Four playing cards, two red and two black, lie facedown on the table. Without looking, you choose any two cards. I’ll bet you $1 that you pick one red card and one black card. Shall we play?

Since you hesitate, I’ll explain the natural fairness of this game. Only two outcomes are possible: the colors match or they do not. Your probability of winning is therefore one-half. The game is fair since neither of us has an advantage. Ready to put your money down?

You won’t be fooled that easily, right? Your sense of caution is admirable. I’ll explain further. Three, not two, outcomes are possible: Both cards you select are red, both are black, or one is black and the other red. The colors match in two of the three cases. Thus, your probability of winning is two-thirds. The game is in your favor. Now will you play?

Your suspicious nature is beginning to annoy me. Very well, here is my final explanation. Four (count ‘em, four) possibilities exist. Consider that the cards are chosen one at a time. Both cards could be red; both could be black; the first red, the second black; or the first black and the second red. The colors match in two of these four possible drawing sequences, giving you (again) a 50-50 chance of winning the bet.

Okay, I’ve listed all possible outcomes in three different analyses, and in every case, your chances of winning the wager are 50-50 or better. So how about that friendly bet?

Did you accept the bet? If so, you have just been hustled. Examine your chances more closely. When you choose the first card, it must be either red or black; the choice has no effect on your chances of winning. Now, of the remaining three cards, how many match the color of the card in your hand? Only one. Therefore, your probability of winning is only one-third (honest, this time!).

So what is wrong with the three previous analyses? To varying degrees, each of them is based on an incomplete listing of all possible outcomes. The incompleteness is subtle and therein lies the hustle.

Denote the four cards as $r_1, r_2, b_1$, and $b_2$. List all possible pairs of cards you can choose: $(r_1, r_2), (r_1, b_1), (r_1, b_2), (r_2, b_1), (r_2, b_2), (b_1, b_2)$. Of the six possible outcomes, only two have matching colors. Your probability of winning is indeed one-third.

Phone Book Follies

Open a telephone book to any page and select any column. In that column, circle any 13 consecutive phone numbers. I will bet $1 that at least two of the phone numbers end in the same two-digit number.

You may feel more confident about making this wager. After all, there are 100 possible two-digit numbers from 00 to 99. Thus, the probability of an exact match would seem to be $1/100$. Even with 13 chances, it would seem unlikely to find a pair of matching two-digit numbers. But again, intuition betrays us.

The sucker in this bet favors the possibility that no match exists among the 13 phone numbers. Let’s analyze the probabilities of that happening. The first number on the list can be any two-digit number and has no effect on the odds. The second number must not match the first; therefore, it must be one of the other 99 two-digit values; that probability is $99/100$. The third number cannot match either of the first two numbers, leaving it 98 possible nonmatching numbers; that probability is $98/100$

The pattern continues down to the thirteenth number, for which only 88 possibilities exist (the other 12 having been taken already); its probability for not matching is $88/100$.

The probability that no match exists among the 13 numbers is the product of the individual chance factors:

$$\frac{99}{100} \times \frac{98}{100} \times \frac{97}{100} \times \cdots \times \frac{88}{100} \approx 0.442775$$

That is, you can expect to lose your bet (no match) about 44.3 percent of the time, and I can expect to win the other 55.7 percent of the time. Simplifying it, my odds of winning are about 5/9. For a truly fair game, I should offer odds: my $5 against your $4, or $1.25 against your $1. Since our original bet was even money (1:1), I have a definite advantage. In 1000 such wagers, I can look forward to winning $557 and losing $443, showing a profit of $114.

Altering the conditions so that you circle 20 numbers increases the probability of a match to 87 percent: 30

(continued)

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people sharing the same birthday is more surprising. Given a gathering of 23 people, the odds that at least two people share the same birthday (month and day) are, surprisingly, better than even (50.7 percent). For 24 people, the probability rises to 53.8 percent. With 41 people, the chance rises to 90.3 percent.

The paradox evaporates when we analyze the situation closely. For simplicity, we exclude February 29 birthdays and assume that birthdays are uniformly distributed among all 365 days in a year. That is, the probability of one's birthday falling on a particular date is exactly 1/365. The probability of two people not sharing a birthday is thus 364/365. Each additional person we consider adds another reducing factor to the probability that no birthdays are shared.

The program in listing 1 lets you explore the probability trends for this general class of sucker bets. You specify the number of possible outcomes; the program shows the probability of a duplicate outcome occurring within a specified number of events.

CARD GAMES

Another class of sucker bets involves playing cards. Here's one of my favorites.

Two shuffled decks are on the table. I bet that among the first six cards in each deck are two identical cards. Your bet, if you choose to accept it, is that no duplicate cards will be found among the first six cards in the two decks. What are your odds?

We draw the first six cards from deck 1. In deck 2, the top card cannot be any of the previously drawn six cards; it must be one of the other 46 cards. The probability of a nonmatch is thus 46/52. The second card must not be any of the six noted cards: it must be one of the other 45 cards of the 51 remaining in deck 2. The probability of a nonmatch is 45/51. The pattern continues for the remaining four cards, giving a cumulative probability of

\[
46 \times 45 \times 44 \times \ldots \times 41 = 0.4600933
\]

Probability of a nonmatch after drawing six cards is about 46 percent, leaving a probability that a match will be drawn of about 54 percent (certainly enough for me to make a living on, if I can find enough suckers to take the bet).

Now suppose we have three shuffled decks on the table. I bet that among the top four cards of each deck will be found two identical cards. You bet that no matches will be found. Are your odds any better this time?

After drawing four cards from deck 1, the odds of not finding a match in four cards from deck 2 are, respectively, 48/52, 47/51, 46/50, and 45/49.

Drawing from deck 3, the first card cannot be any of the eight previously drawn cards; probability of that is 44/52. Probabilities for the next three cards not matching are 43/51, 42/50, and 41/49.

Multiplying the probabilities gives a
Cumulative probability for not finding a match of 0.3603997, or about 36 percent. My chances for winning the bet are thus around 64 percent. I can live like a king on that.

Given D decks, each containing N cards, this program calculates the probability of finding at least two identical cards among the top C cards of each deck.

How many decks (D) ? 3
How many cards in each deck (N) ? 52
How many cards from each deck (1- 52) ? 4
Cumulative probability of a MATCH= .6396003

Matthew Hall
Mathematical Recreations

10 PRINT "Given D decks, each containing N cards,"
20 PRINT "this program calculates the probability"
30 PRINT "of finding at least two identical cards"
40 PRINT "among the top C cards of each deck."
50 PRINT
60 INPUT "How many decks (>1)"; D
70 IF D<2 THEN END
80 INPUT "How many cards in each deck (>0)"; N
90 IF N<1 THEN END
100 PRINT "Draw how many cards from each deck (1-" ; N; ");"
110 INPUT C
120 IF C<1 OR C>N THEN END
130 PRINT "Successive chances for NOT matching..."
140 P=1
150 FOR J=1 TO D-1
160 FOR I=1 TO C
170 X=N+1-J*C-I
180 Y=N+1-I
190 PRINT USING "### / ### = #.#####; X,Y,X/Y"
200 P=P*Y/X
210 IF P<0 THEN I=C: J=D-1
220 NEXT J
230 NEXT I
240 PRINT "Cumulative probability of a MATCH=":1-P